APPROXIMATION OF LYAPUNOV EXPONENTS IN NON-ARCHIMEDEAN AND COMPLEX DYNAMICS

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Abstract. We give two kinds of approximation of Lyapunov exponents of rational functions of degree more than one on the projective line over more general fields than that of complex numbers.

1. Introduction

Let K be an algebraically closed field complete with respect to a non-trivial absolute value $|\cdot|$. The field K is said to be non-archimedean if the strong triangle inequality $|z-w| \le \max\{|z|,|w|\}$ $(z, w \in K)$ holds (e.g. p-adic $\mathbb{C}_p$). Otherwise, K is archimedean, and then indeed $K \cong \mathbb{C}$. We always assume that K has characteristic 0. The Berkovich projective line $\mathsf{P}^1 = \mathsf{P}^1(K)$ produces a compactification of the (classical) projective line $\mathbb{P}^1 = \mathbb{P}^1(K)$. For archimedean K, P^1 and $\mathbb{P}^1$ are identical. For the details of potential theory and dynamics on P^1, see [1], [4], [5].

Let f be a rational function on $\mathbb{P}^1$ of degree $d > 1$. The action of f on $\mathbb{P}^1$ extends to a continuous, open, surjective and (fiber-)discrete map on P^1, preserving $\mathbb{P}^1$ and $\mathsf{P}^1 \setminus \mathbb{P}^1$. The exceptional set $E(f)$ of (the extended) f is the set of all points $a \in \mathbb{P}^1$ such that $\#\bigcup_{k\in\mathbb{N}} f^{-k}(a) < \infty$.

For a rational function (say, possibly moving target) a on $\mathbb{P}^1$ and each $k \in \mathbb{N}$, there are exactly $d^k + \deg a$ roots of the equation $f^k = a$ in $\mathbb{P}^1$ counting their multiplicity, unless $f^k \not\equiv a$. Let δ_w be the Dirac measure at $w \in \mathsf{P}^1$. Let us consider the averaged distribution

$$\nu_k^a := \frac{1}{d^k + \deg a} \sum_{w \in \mathbb{P}^1 : f^k(w) = a(w)} \delta_w$$

of roots of $f^k = a$ in $\mathbb{P}^1$, where the sum takes into account the multiplicity of each root. Let Δ be the normalized Laplacian on P^1. The equilibrium (or canonical) measure of f on P^1 is the Radon probability

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measure

$$\mu_f := \Delta g_f + \Omega_{\mathrm{can}}$$

on P^1, where the continuous function g_f is the dynamical Green function of f on P^1 (cf. [7, §2]), and Ω_{can} denotes the normalized Fubini-Study area element on $\mathbb{P}^1$ for archimedean K, and the Dirac measure at the canonical (or Gauss) point $\mathcal{S}_{\mathrm{can}}$ on P^1 for non-archimedean K.

The equidistribution theorem for possibly moving targets, which is in complex dynamics by Lyubich [6, Theorem 3] and in non-archimedean dynamics by Favre and Rivera-Letelier [4, Theorèmes A et B], is

Theorem 1.1 ([6, Theorem 3], [4, Théorèmes A et B])**.** *Let f be a rational function on $\mathbb{P}^1 = \mathbb{P}^1(K)$ of degree > 1. If a rational function a on $\mathbb{P}^1$ does not identically equal a value in $E(f)$, then $\nu_k^a \to \mu_f$ weakly on P^1 as $k \to \infty$.*

Let $f^\#$ be the chordal derivative of f with respect to the normalized chordal distance $[\cdot,\cdot]$ on $\mathbb{P}^1$. Then the function $f^\#$ extends continuously to P^1. We define the Lyapunov exponent of f with respect to μ_f by

$$L(f) := \int_{\mathsf{P}^1} \log(f^\#)\mathrm{d}\mu_f > -\infty$$

(cf. [8, formula (1.2)]). The (classical) critical set $C(f)$ of f is

$$C(f) := \{c \in \mathbb{P}^1; f^\#(c) = 0\}.$$

A point in the orbits of some periodic $c \in C(f)$ under f is called a superattracting periodic point of f.

We give two kinds of approximation of $L(f)$.

Repelling periodic points. A periodic point $p \in \mathbb{P}^1$ of f of period $k \in \mathbb{N}$ is said to be repelling if $(f^k)^\#(p) = |(f^k)'(p)| > 1$. For each $k \in \mathbb{N}$, set

$$R_k(f) := \{p \in \mathbb{P}^1;\ \text{repelling periodic points of } f \text{ of period } k\},$$

$$\nu_k^{\mathrm{rep}} := \frac{1}{d^k+1} \sum_{w \in R_k(f)} \delta_w,$$

$$R_k^*(f) := R_k(f) \setminus \bigcup_{j \in \mathbb{N}; j<k, j|k} R_j(f), \quad \nu_k^* := \frac{1}{d^k+1} \sum_{w \in R_k^*(f)} \delta_w.$$

From an argument based on Bezout's theorem, for every $k \in \mathbb{N}$,

$$\#\left(\bigcup_{j \in \mathbb{N}; j<k, j|k} R_j(f)\right) \le 2kd^{k/2} = o(d^k) \tag{1.1}$$

as $k \to \infty$ (cf. [3, §4.2]). If there are at most finitely many non-repelling periodic points of f in $\mathbb{P}^1$, then Theorem 1.1 together with (1.1) implies

the equidistribution

$$\nu_k^{\mathrm{rep}} \to \mu_f \quad \text{and} \quad \nu_k^* \to \mu_f \tag{1.2}$$

weakly on P^1 as $k \to \infty$. In particular, for archimedean K, from Fatou's finiteness on non-repelling periodic points of f, (1.2) always holds.

The following is a consequence of [8, Theorem 1], which is a generalization of [3] for archimedean fields (see also [2]) and of [10] for number fields or function fields to general K. The finiteness assumption is vacuous for archimedean K:

Theorem 1. *Let f be a rational function on $\mathbb{P}^1 = \mathbb{P}^1(K)$ of degree $d > 1$. If there are at most finitely many non-repelling periodic points of f in $\mathbb{P}^1$, then*

$$\lim_{k\to\infty} \frac{1}{d^k+1} \sum_{w \in R_k(f)} \frac{1}{k} \log (f^k)^{\#}(w) = L(f), \tag{1.3}$$

$$\lim_{k\to\infty} \frac{1}{d^k+1} \sum_{w \in R_k^*(f)} \frac{1}{k} \log (f^k)^{\#}(w) = L(f), \tag{1.4}$$

and $L(f) \geq 0$.

Since this important case could be shown by a simpler argument than that in [8, §4] and (1.4) is not mentioned there, we give herewith a (simpler) proof of Theorem 1. The proof of Theorem 1 for archimedean K in [3] was also based on the fact that $L(f)(\geq \log \sqrt{d}) > 0$ holds for archimedean K (but this does not always hold for non-archimedean K).

Preimages of points. Put $B[z, r] := \{w \in \mathbb{P}^1; [w, z] < r\}$ for each $z \in \mathbb{P}^1$ and each $r > 0$. Under the action f on $\mathbb{P}^1$, a point $z_0 \in \mathbb{P}^1$ is said to be *wandering* if $\#\{f^k(z_0); k \in \mathbb{N} \cup \{0\}\} = \infty$. Let $C(f)_{\mathrm{wan}}$ be the set of all $c \in C(f)$ wandering under f. The subset

$$E^{1/2}_{\mathrm{wan}}(f) := \bigcup_{c \in C(f)_{\mathrm{wan}}} \bigcap_{N \in \mathbb{N}} \bigcup_{j \geq N} B[f^j(c), \exp(-d^{j/2})]$$

is of finite Hyllengren measure for the increasing sequence $(d^{j/2}) \subset \mathbb{N}$, so of (chordal) capacity 0 (for a direct proof, see [7, Lemma 2.1]). Set

$$E_{\mathrm{Lyap}}(f) := \left\{ a \in \mathbb{P}^1; \int_{\mathsf{P}^1} \log(f^{\#}) \mathrm{d}\nu_k^a \not\to L(f) \text{ as } k \to \infty \right\}.$$

This contains the finite set $\{f^j(c); c \in C(f) \setminus C(f)_{\mathrm{wan}}, j \in \mathbb{N}\}$ of all orbits of non-wandering (or preperiodic) critical points of f.

Theorem 2. *Let f be a rational function on $\mathbb{P}^1 = \mathbb{P}^1(K)$ of degree $d > 1$. Then*

$$E_{\mathrm{Lyap}}(f) \subset E^{1/2}_{\mathrm{wan}}(f) \cup \{f^j(c); c \in C(f) \setminus C(f)_{\mathrm{wan}}, j \in \mathbb{N}\}.$$

In particular, for every $a \in \mathbb{P}^1$ except for a set of capacity 0*,*

$$\lim_{k\to\infty} \frac{1}{d^k} \sum_{w \in f^{-k}(a)} \log(f^\#)(w) = L(f),$$

where the sum takes into account the multiplicity of each root w of the equation $f(\cdot) = a$ on $\mathbb{P}^1$.

2. Background

Let f be a rational function on $\mathbb{P}^1 = \mathbb{P}^1(K)$ of degree $d > 1$. We fix a projective coordinate on $\mathbb{P}^1$ so that $K \cong \mathbb{P}^1 \setminus \{\infty\}$. Let $[\cdot, \cdot]$ be the chordal distance on $\mathbb{P}^1$ normalized as $[0, \infty] = 1$ (for the precise definition, see, for example, [8, §2]).

For non-archimedean K, a typical element $\mathcal{S} \in \mathsf{P}^1 = \mathsf{P}^1(K)$ other than ∞ is regarded as a (K-closed) disk $\{z \in K; |z - a| \le r\}$ for some $a \in K$ and some $r =: \mathrm{diam}(\mathcal{S}) \ge 0$. The point $\mathcal{S}_{\mathrm{can}} := \{z \in K; |z| \le 1\} \in \mathsf{P}^1$ is called the canonical (or Gauss) point in P^1. For disk $\mathcal{S}$, put $|\mathcal{S}| := \sup_{z \in \mathcal{S}} |z|$. For disks $\mathcal{S}, \mathcal{S}' \in \mathsf{P}^1$, let $\mathcal{S} \wedge \mathcal{S}'$ be the smallest disk containing $\mathcal{S} \cup \mathcal{S}'$.

For non-archimedean K, $[\cdot, \cdot]$ canonically extends to the generalized Hsia kernel $[\mathcal{S}, \mathcal{S}']_{\mathrm{can}}$ on P^1 with respect to $\mathcal{S}_{\mathrm{can}}$ satisfying for example that for disks $\mathcal{S}, \mathcal{S}' \in \mathsf{P}^1$,

$$[\mathcal{S}, \mathcal{S}']_{\mathrm{can}} = \frac{\mathrm{diam}(\mathcal{S} \wedge \mathcal{S}')}{\max\{1, |\mathcal{S}|\} \max\{1, |\mathcal{S}'|\}}.$$

This extension is separately continuous on each variable $\mathcal{S}, \mathcal{S}' \in \mathsf{P}^1$, and vanishes if and only if $\mathcal{S} = \mathcal{S}' \in \mathbb{P}^1$. Let us also denote this extension by the same $[\cdot, \cdot]$, for simplicity.

For general K, the notion of (chordal) capacity of a Borel set in P^1 is introduced as usual using the chordal kernel $\log[\cdot, \cdot]$ on P^1. We note that a countable set in $\mathbb{P}^1$ is of capacity 0. The chordal potential of a Radon measure μ on P^1 is defined as

$$U^\#_\mu(\cdot) := \int_{\mathsf{P}^1} \log[\cdot, \mathcal{S}] \mathrm{d}\mu(\mathcal{S})$$

on P^1. We recall the following lemmas.

Lemma 2.1 ([8, Lemma 3.1])**.** *Let f be a rational function on $\mathbb{P}^1 = \mathbb{P}^1(K)$ of degree $d > 1$. Suppose that a sequence (ν_k) of positive Radon measures on P^1 tends to μ_f weakly on P^1 as $k \to \infty$. Then*

$$\lim_{k\to\infty} \int_{\mathsf{P}^1} \log f^\# \mathrm{d}\nu_k = L(f)$$

holds if for each $c \in C(f)$, $\lim_{k\to\infty} U^\#_{\nu_k}(c) = U^\#_{\mu_f}(c)$.

Lemma 2.2 ([8, Lemma 3.3]). *Let f be a rational function on $\mathbb{P}^1 = \mathbb{P}^1(K)$ of degree $d > 1$. Let a be a rational function on $\mathbb{P}^1$ which does not identically equal a value in $E(f)$, and let (S_k) be a sequence of subsets $S_k \subset \mathsf{P}^1$. Then for every $z \in \mathbb{P}^1 \setminus (\limsup_{k\to\infty} \operatorname{supp} \nu_k^a)$,*

$$U^{\#}_{\nu_k^a|(\mathsf{P}^1\setminus S_k)}(z) - U^{\#}_{\mu_f}(z) = \frac{1}{d^k + \deg a}\log[f^k(z), a(z)] - U^{\#}_{\nu_k^a|S_k}(z) + o(1)$$

as $k \to \infty$.

The Berkovich Julia set $\mathsf{J}(f)$ is defined by the set of all $\mathcal{S} \in \mathsf{P}^1$ satisfying

$$\bigcap_{U\text{: an open neighborhood of } \mathcal{S} \text{ in } \mathsf{P}^1} \left(\bigcup_{k\in\mathbb{N}} f^k(U)\right) = \mathsf{P}^1 \setminus E(f)$$

([4, Definition 2.8]). The Berkovich Fatou set $\mathsf{F}(f)$ is $\mathsf{P}^1 \setminus \mathsf{J}(f)$, which is open in P^1. We note that any superattracting (resp. repelling) periodic points of f is in $\mathsf{F}(f)$ (resp. in $\mathsf{J}(f)$).

3. A proof of Theorem 1

Set $a = \mathrm{Id}_{\mathbb{P}^1}$ and $S_k = \{w \in \mathbb{P}^1; f^k(w) = w\} \setminus R_k(f)$ for each $k \in \mathbb{N}$. Then $\nu_k^{\mathrm{rep}} = \nu_k^a|(\mathsf{P}^1 \setminus S_k)$. As seen in Introduction, under the assumptions in Theorem 1, the equidistribution (1.2) holds. In particular,

$$\lim_{k\to\infty} \nu_k^a|S_k = 0 \tag{3.1}$$

weakly on P^1.

Let us show that $\lim_{k\to\infty} U^{\#}_{\nu_k^{\mathrm{rep}}}(c) = U^{\#}_{\mu_f}(c)$ for each $c \in C(f)$. Then Lemma 2.1 will conclude that

$$\lim_{k\to\infty} \int_{\mathsf{P}^1} \log(f^{\#}) \mathrm{d}\nu_k^{\mathrm{rep}} = L(f), \tag{3.2}$$

which with the chain rule shows (1.3).

Let $c \in C(f) \cap \mathsf{J}(f)$. Then c is not periodic under f, or equivalently, $c \in \mathbb{P}^1 \setminus (\bigcup_{k\in\mathbb{N}} \operatorname{supp} \nu_k^a)$. In particular, $c \notin \bigcup_{k\in\mathbb{N}} S_k$, and under the assumption in Theorem 1, which is equivalent to $\#\bigcup_{k\in\mathbb{N}} S_k < \infty$, we have $\inf\{[c, w]; w \in \bigcup_{k\in\mathbb{N}} S_k\} > 0$. This with (3.1) implies that

$$\lim_{k\to\infty} U^{\#}_{\nu_k^a|S_k}(c) = 0.$$

Moreover, Przytycki [9, Lemma 1] asserts that *for any $c \in C(f) \cap \mathsf{J}(f)$ and every $k \in \mathbb{N}$,*

$$[f^k(c), c] \geq \frac{1}{10}(\max\{1, \sup_{\mathsf{P}^1} f^{\#}\})^{-k+1}$$

(the original proof of [9, Lemma 1] for archimedean K works for non-archimedean K), so for every $c \in C(f) \cap \mathsf{J}(f)$,

$$\lim_{k\to\infty} \frac{1}{d^k+1} \log[f^k(c), c] = 0.$$

Hence Lemma 2.2 implies that $\lim_{k\to\infty} U^{\#}_{\nu_k^{\mathrm{rep}}}(c) = U^{\#}_{\mu_f}(c)$.

Next, let $c \in \mathsf{F}(f)$ (and we will not use Lemma 2.2). Then $\log[c, \cdot]$ is continuous, i.e, does not take $-\infty$, on $\mathsf{J}(f)$. Since $\bigcup_{k\in\mathbb{N}} R_k(f) \subset \mathsf{J}(f)$, the equidistribution (1.2) implies that $\lim_{k\to\infty} U^{\#}_{\nu_k^{\mathrm{rep}}}(c) = U^{\#}_{\mu_f}(c)$.

Now the proof of (3.2) is complete.

Finally, from $1 \le \inf_{\bigcup_k R_k(f)} f^{\#} \le \sup_{\mathbb{P}^1} f^{\#} < \infty$ and (1.1), we have

$$\lim_{k\to\infty} \int_{\mathsf{P}^1} \log f^{\#} \mathrm{d}\nu_k^* = \lim_{k\to\infty} \int_{\mathsf{P}^1} \log f^{\#} \mathrm{d}\nu_k^{\mathrm{rep}}$$

(if one of the limits exists), which together with (3.2) and the chain rule shows (1.4). Now the last assertion $L(f) \ge 0$ is obvious. $\square$

4. A proof of Theorem 2

We set $S_k = \emptyset$ for each $k \in \mathbb{N}$. Let $a \in \mathbb{P}^1 \setminus (E^{1/2}_{\mathrm{wan}}(f) \cup \{f^j(c); c \in C(f) \setminus C(f)_{\mathrm{wan}}, j \in \mathbb{N}\})$. Then for every $c \in C(f)$, we have $c \in \mathbb{P}^1 \setminus (\limsup_{k\to\infty} \operatorname{supp} \nu_k^a)$ and

$$\liminf_{k\to\infty} \frac{1}{d^k} \log[f^k(c), a] \ge \liminf_{k\to\infty} (-d^{-k/2}) = 0,$$

which with $[\cdot, \cdot] \le 1$ implies that $\lim_{k\to\infty} (\log[f^k(c), a])/d^k = 0$. From this and the assumption that $S_k = \emptyset$ for each $k \in \mathbb{N}$, Lemma 2.2 implies that $\lim_{k\to\infty} U^{\#}_{\nu_k^a}(c) = U^{\#}_{\mu_f}(c)$. From this, Lemma 2.1 completes the proof of Theorem 2. $\square$

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